Spaces of 2D classical and generalized elastic materials: a geometric journey

IRP Coss&Vita, M&MoCS, F2M

Nicolas Auffray [1] Giuseppe Rosi [1] Luca Placidi [2]

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October 17th - Kick-off meeting

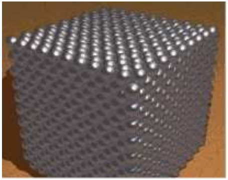
Outline

Architectured materials

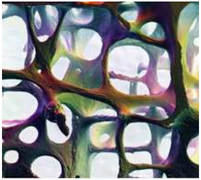
Definition

A material will be said to be architectured if:

- It presents, between its microstructure and its macrostructure, one or more other scales of organization of matter;
- If the intermediate organization scales are commensurable with those of the microstructure and/or the macrostructure.



(a) Stacking spheres



(b) Trabecular bone

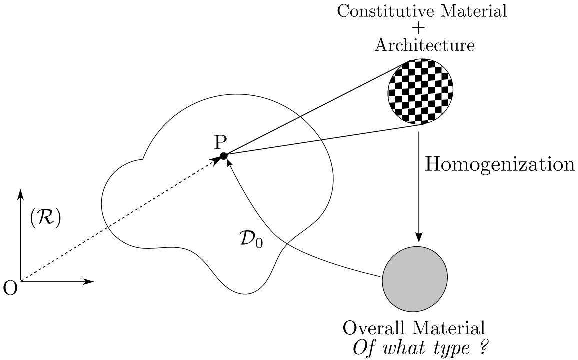
(c) Coextruded steel

Characteristics of architectured materials

- Multi-functional applications and multi-physical behaviours;
- Strong anisotropy;
- Weak separation between the different scales of the material.

CONTINUOUS DESCRIPTION OF STRUCTURAL EFFECTS

We want to make architecture disappear ...



...while maintaining structural effects at the material level.

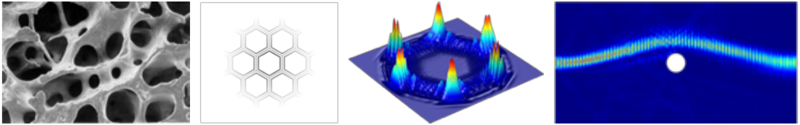
The complexity is contained in the algebraic structure of the constitutive law.

$\Rightarrow$ **How to make this transition rigorously is the job of the UP research group (c.f. A. Lebée).**

Elasto-Dynamics of Microstructured Media (ELADYN)

- **Main topics:**
  1. Theoretical framework for anisotropic generalized continua;
  2. Wave propagation in microstructured media;
  3. Continuum simulation of wave propagation in mechanical metamaterials;
  4. Development of experimental testing devices adapted to architectured materials.

- **Coordinators:**
  - **French side:** Nicolas Auffray, Giuseppe Rosi;
  - **Italian side:** Luca Placidi.



(a) Trabecular bone (b) Honeycomb (c) Energy flow in honeycomb (d) Wave control

(SELECTED) PAST ACTIONS (2015-2018)

- Summer School on Elastic metamaterials, Alghero, Italy, 22-29th of May 2016 (with G. Milton)

- Exchange of researchers: around 20 weeks of exchanges were funded between France and Italy for the former META R.G.

Product of the Research Group

- Publications: Around 10 publications involving at least one French and one Italian author belonging to F2M and M&MoCS;
- Co-advised PhD: Mario Spagnuolo:*Continuous models for multi-phase architectured/meta materials* Directors: P. Franciozi (LSPM,UP13), F. dell'Isola (La Sapienza,Roma);
- National contracts:
  - **METAMORPH (PEPS INSIS)**: Caractérisation mécanique inverse des métamatériaux: modélisation, identification expérimentale des paramètres et évolutions possibles. *Coordinator*: J. Réthoré (INSA LYON);
  - **Strain gradient materials vs. stress gradient materials (PEPS INSIS)**. *Coordinator*: K. Sab (Navier, Marne-la-Vallée);
  - **ANR ArchiMatHOS** (2018-2022): Design of architectured materials using higher order homogenization. *Coordinator*: A. Lebée (Navier, Marne-la-Vallée);
  - **ANR MoMaP** (2019-2023): Mesure et Optimisation des Matériaux Architecturés Périodiques *Coordinator*: M. Francois (GeM, Université de Nantes);
  - **ANR Max-Oasis** (2020-2024): Matériaux Architecturés eXotiques, Ondes, AniSotropie, InStabilités. *Coordinator*: N. Auffray (MSME, Marne-la-Vallée);

Spaces of 2D classical and generalized elastic materials

Aims of the talk

1. set the differences between elastic tensors and elastic materials;
2. describe the elastic material domain in terms of invariants of the integrity basis;
3. extend the approach to generalized continuum elasticity;

The 2D setting

1. complex enough to produce non trivial results;
2. simple enough to handle explicit computations;
3. illustrate the possibilities of the geometric approach;
4. construct situations that can be extended to the 3D problem.

Outline

THE ELASTICITY TENSOR

Hooke's law

Linear relation between the stress tensor $\underset{\sim}{\sigma} \in S^2(\mathbb{R}^d)$ and the strain tensor $\underset{\sim}{\varepsilon} \in S^2(\mathbb{R}^d)$:

$$\underset{\sim}{\sigma} = \underset{\approx}{\mathrm{C}} : \underset{\sim}{\varepsilon}$$

with $\underset{\approx}{\mathrm{C}}$ an element of the vector space $\mathbb{E}\mathrm{la}_4(d) := S^2(S^2(\mathbb{R}^d))$;

O(d)-action

O(d) acts on $\mathbb{E}\mathrm{la}_4(d)$ through $\star$ defined by:

$$\star : \mathrm{O}(d) \times \mathbb{E}\mathrm{la}_4(d) \to \mathbb{E}\mathrm{la}_4(d) \; ; \; (\underset{\sim}{\mathrm{Q}}, \underset{\approx}{\mathrm{C}}) \mapsto \underset{\sim}{\mathrm{Q}} \star \underset{\approx}{\mathrm{C}} := Q_{io}Q_{jp}Q_{kq}Q_{lr}C_{opqr}$$

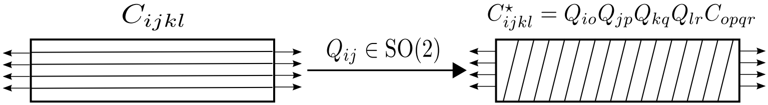


Figure: Different samples extracted from the same material.

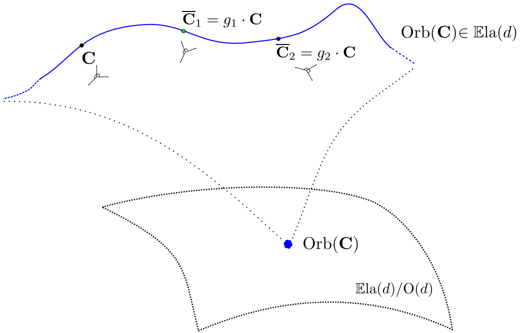
AN ELASTIC MATERIAL: A $\mathrm{O}(d)$ -ORBIT

Orbit

The set of tensors of $\mathbb{E}\mathrm{la}_4(d)$ $\mathrm{O}(d)$-conjugate to $\underset{\approx}{\mathrm{C}}$ constitutes its $\mathrm{O}(d)$-orbit :

$$\mathrm{Orb}(\underset{\approx}{\mathrm{C}}) := \left\{ \overline{\underset{\approx}{\mathrm{C}}} = \mathrm{Q} \star \underset{\approx}{\mathrm{C}} \mid \mathrm{Q} \in \mathrm{O}(d) \right\}.$$

The orbits space is the quotient space $\mathbb{E}\mathrm{la}_4(d)/\mathrm{O}(d)$.



Note

An elastic material is

- An orbit of $\mathbb{E}\mathrm{la}_4(d)$;
- A point of $\mathbb{E}\mathrm{la}_4(d)/\mathrm{O}(d)$.

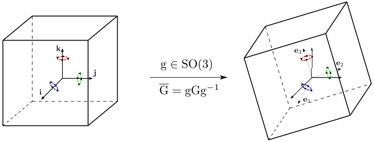
SYMMETRY PROPERTIES

Symmetry Group

Symmetry group of an elasticity tensor:

$$G_{\underset{\approx}{C}} := \left\{ \underset{\sim}{Q} \in O(d), \quad \underset{\sim}{Q} \star \underset{\approx}{C} = \underset{\approx}{C} \right\}.$$

Tensors on the same orbit have conjugate symmetry groups.



Groupe du cube d'axes
i, **j**, **k**

Groupe du cube d'axes
$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$

Symmetry Class

The class of symmetry is the conjugacy class of a symmetry group.

$$[G_{\underset{\approx}{C}}] := \left\{ \underset{\sim}{Q} G_{\underset{\approx}{C}} \underset{\sim}{Q}^{-1}, \quad \underset{\sim}{Q} \in O(d) \right\}.$$

$\mathbb{E}\mathrm{la}_4(d)$ is divided into strata of different symmetry classes.

2D Elasticity tensors

Symmetry groups of 2D elasticity tensors belong to 4 symmetry classes:

	Biclinic	Orthotropic	Tetragonal	Isotropic
$[G_{\underset{\approx}{C}}]$	$[Z_2]$	$[D_2]$	$[D_4]$	$[O(2)]$
$\#_{\text{indep}}(\underset{\approx}{C})$	6 (5)	4	3	2

Figure: Schematic figures of the symmetry classes of 2D elasticity tensors

The space of elasticity tensors is divided into 4 strata:

$$\mathbb{E}\text{la}_4 = \Sigma_{[Z_2]} \cup \Sigma_{[D_2]} \cup \Sigma_{[D_4]} \cup \Sigma_{[O(2)]}$$

HARMONIC DECOMPOSITION IN $\mathbb{R}^2$

Definition

Let $\mathbb{K}^n$ be the space of nth-order harmonic tensors in 2D, its elements are:

1. n-th order tensors;
2. symmetric with respect to the permutation of all the indices;
3. traceless.

In $\mathbb{R}^2$, we have

$$\dim \mathbb{K}^n = \begin{cases} 2, & n \geq 1 \\ 1, & n = 0, -1 \end{cases}$$

Transformation of irreducible components

For all $n \geq 1$, O(2)-action on $\mathbb{K}^n$ is given by ρ_n:

$$\rho_n(\underset{\sim}{\mathrm{R}}(\theta)) := \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}, \quad \rho_n(\underset{\sim}{\mathrm{P}}(\underline{\mathrm{e}}_2)) := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

BASIS FOR O(2)-POLYNOMIAL INVARIANTS OF $\mathbb{E}\mathrm{la}_4$

$\underset{\approx}{\mathrm{C}}$ can be decomposed as follows

$$\underset{\approx}{\mathrm{C}} = \underset{\approx}{\mathrm{D}} + \frac{1}{2}(\underset{\sim}{1} \otimes \underset{\sim}{\mathrm{d}} + \underset{\sim}{\mathrm{d}} \otimes \underset{\sim}{1}) + \frac{\kappa}{2}\underset{\approx}{\mathrm{P}}^0 + \frac{\gamma}{2}\underset{\approx}{\mathrm{P}}^2$$

with

$$\underset{\approx}{\mathrm{D}} \in \mathbb{K}^4, \quad \underset{\approx}{\mathrm{d}} \in \mathbb{K}^2, \quad \kappa, \gamma \in \mathbb{K}^0,$$

O(2)-integrity basis of $\mathbb{E}\mathrm{la}_4$

The following quantities

$$I_1 = \kappa, \quad J_1 = \gamma, \quad I_2 = \underset{\sim}{\mathrm{d}} : \underset{\sim}{\mathrm{d}}, \quad J_2 = \underset{\approx}{\mathrm{D}} :: \underset{\approx}{\mathrm{D}}, \quad I_3 = \underset{\sim}{\mathrm{d}} : \underset{\approx}{\mathrm{D}} : \underset{\sim}{\mathrm{d}}$$

- A list $(I_1, J_1, I_2, J_2, I_3)$ specifies uniquely an elastic material;
- Any O(2) polynomial invariant of $\mathbb{E}\mathrm{la}_4$ is a polynomial in $(I_1, J_1, I_2, J_2, I_3)$;
- Verify the inequality (Cauchy-Schwarz): $I_2^2 J_2 - 2I_3^2 \geq 0$;

GEOMETRIC DOMAIN OF THE ORBIT SPACE [4]

Biclinic: $\Sigma_{[Z_2]}$

$\Big\downarrow$ $I_2^2 J_2 - 2I_3^2 = 0,\ J_2 \neq 0$

Orthotropic: $\Sigma_{[D_2]}$

$\Big\downarrow$ $I_2 = 0$

Tetragonal: $\Sigma_{[D_4]}$

$\Big\downarrow$ $J_2 = 0$

Isotropic: $\Sigma_{[O(2)]}$

Figure: Breaking conditions between strata

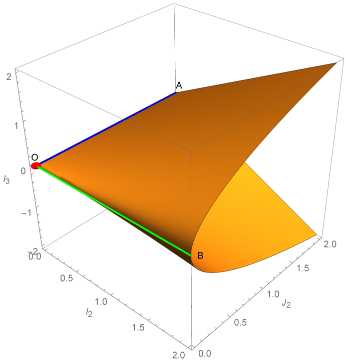


Figure: Algebraic variety of elastic materials with respect to (I_2, J_2, I_3)

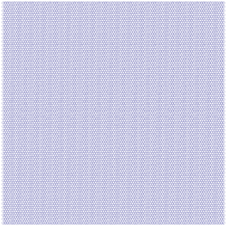
This situation is exceptional in its simplicity

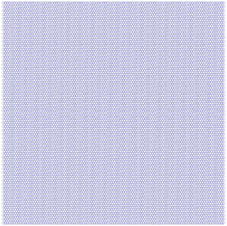
OUTLINE

HEXAGONAL ANISOTROPY

Elastodynamic (with G. Rosi)

Experiment: Propagation of elastic waves in a hexagonal lattice [6, 8]





Observation: At low frequency, the propagation is isotropic, when the frequency increases the propagation becomes hexagonal ...

Hexagonal anisotropy

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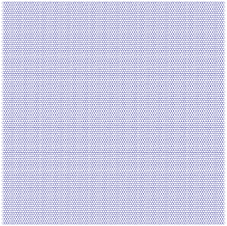

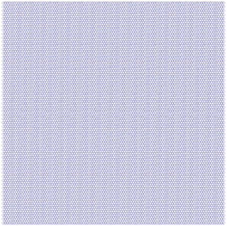

Observation: At low frequency, the propagation is isotropic, when the frequency increases the propagation becomes hexagonal ...

$\Rightarrow$ **The classical elasticity does not see the hexagonal anisotropy**

STRAIN-GRADIENT ELASTICITY: GENERAL CASE

Degrees of freedom: DDL $= \{\underline{u}\}$; $\underline{u} \in \mathbb{R}^d$
State variables associated with the kinematics

$$\text{PSV} = \{\underset{\sim}{\varepsilon}, \underset{\sim}{\varepsilon} \otimes \underline{\nabla}\}$$

Linear constitutive law **(coupled)**:

$$\begin{cases} \underset{\sim}{\sigma} = \underset{\approx}{C} : \underset{\sim}{\varepsilon} + \underset{\approx}{M} \therefore \underset{\simeq}{\eta} \\ \underset{\simeq}{\tau} = \underset{\approx}{M}^T : \underset{\sim}{\varepsilon} + \underset{\approx}{A} \therefore \underset{\simeq}{\eta} \end{cases}$$

- $\underset{\sim}{\varepsilon}$: strain tensor;
- $\underset{\simeq}{\eta} = \underset{\sim}{\varepsilon} \otimes \underline{\nabla}$: strain gradient tensor;
- $\underset{\sim}{\sigma}$: Cauchy stress tensor;
- $\underset{\simeq}{\tau}$: hyperstress tensor.

New elasticity tensors:

- $\underset{\approx}{M}$ allows coupling in non-centro symmetric materials (order ϵ^1) [2, 5];
- $\underset{\approx}{A}$ allows hexatropic wave propagation (order ϵ^2) [6, 8, 7].

STRAIN-GRADIENT ELASTICITY: CENTRO-SYMMETRIC CONTINUUM

Degrees of freedom: DDL $= \{\underline{u}\} \quad ; \quad \underline{u} \in \mathbb{R}^d$
State variables associated with the kinematics

$$\text{PSV} = \{\underset{\sim}{\varepsilon}, \underset{\sim}{\varepsilon} \otimes \underline{\nabla}\}$$

Linear constitutive law **(uncoupled)**:

$$\begin{cases} \underset{\sim}{\sigma} = \underset{\approx}{C} : \underset{\sim}{\varepsilon} \\ \underset{\simeq}{\tau} = \underset{\approx}{A} \therefore \underset{\simeq}{\eta} \end{cases}$$

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Anisotropic properties of $\mathbb{E}\mathrm{la}_6$

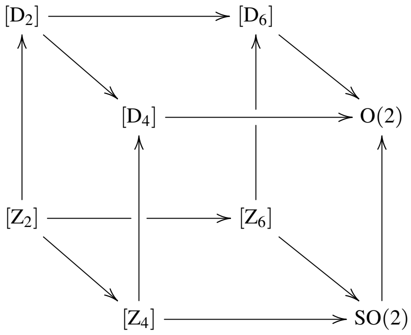
Space of 6-th order tensors:

$$\mathbb{E}\mathrm{la}_6 := \{\underset{\approx}{\mathrm{A}} \in \otimes^6(\mathbb{R}^2) | A_{\underline{(ij)k}\ \underline{(lm)n}}\}$$

Symmetry classes [1, 3]

$$\mathfrak{I}(\mathbb{E}\mathrm{la}_6) = \{[\mathrm{Z}_2], [\mathrm{D}_2], [\mathrm{Z}_4], [\mathrm{D}_4], [\mathrm{Z}_6], [\mathrm{D}_6], [\mathrm{SO}(2)], [\mathrm{O}(2)]\}$$

Bifurcation diagram:



New Features:

- Chiral sensitivity;
- Higher order anisotropy.

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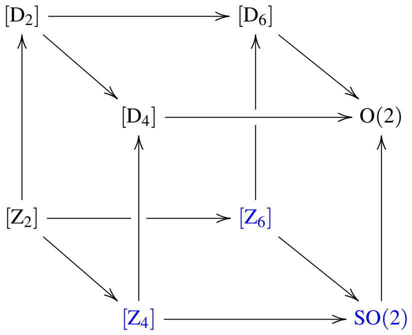

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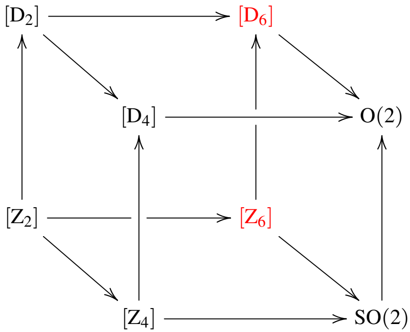

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The harmonic structure of $\mathbb{E}\mathrm{la}_6$

$$\mathbb{E}\mathrm{la}_6 \simeq \mathbb{K}^6 \oplus 2\mathbb{K}^4 \oplus 5\mathbb{K}^2 \oplus 3\mathbb{K}^0 \oplus \mathbb{K}^{-1}$$

Dimensions of the anisotropic operators

Nom	Digonale	Orthotrope	Tetrachirale	Tetragonale
$[\mathrm{G}_{\underset{\approx}{\mathrm{A}}}]$	$[\mathrm{Z}_2]$	$[\mathrm{D}_2]$	$[\mathrm{Z}_4]$	$[\mathrm{D}_4]$
$\#_{\mathrm{indep}}(\underset{\approx}{\mathrm{A}})$	21 (20)	12	9 (8)	6

Nom	Hexachirale	Hexagonale	Hemitrope	Isotrope
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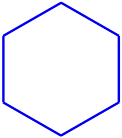
The hexatropic situation: D_6

$$\begin{pmatrix} \underset{\sim}{\sigma} \\ \underset{\simeq}{\tau} \end{pmatrix} = \begin{pmatrix} \underset{\approx}{\mathrm{C}}^{\mathrm{O}(2)} & 0 \\ 0 & \underset{\approx}{\mathrm{A}}^{\mathrm{D}_6} \end{pmatrix} \begin{pmatrix} \underset{\sim}{\varepsilon} \\ \underset{\simeq}{\eta} \end{pmatrix} \qquad \text{avec } \underset{\simeq}{\eta} = \underset{\sim}{\varepsilon} \otimes \underline{\nabla}$$



with:

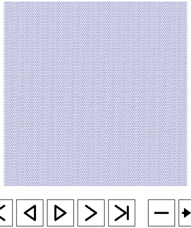
$$\underset{\approx}{\mathrm{C}}^{\mathrm{O}(2)} = \begin{pmatrix} c_{11} & c_{12} & 0 \\ & c_{11} & 0 \\ & & c_{11} - c_{12} \end{pmatrix};$$

and

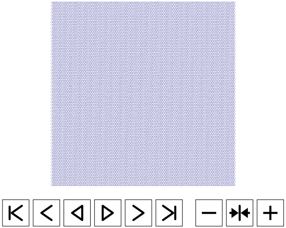
$$\underset{\approx}{\mathrm{A}}^{\mathrm{D}_6} = \begin{pmatrix} a_{11} & a_{12} & \frac{a_{11}-a_{22}}{\sqrt{2}} - a_{23} & 0 & 0 & 0 \\ & a_{22} & a_{23} & 0 & 0 & 0 \\ & & \frac{a_{11}+a_{22}}{2} - a_{12} & 0 & 0 & 0 \\ & & & a_{44} & a_{11}-a_{44}+a_{12} & \frac{3a_{11}-a_{22}}{\sqrt{2}} - a_{23} - \sqrt{2}a_{44} \\ & & & & a_{22}+a_{44}-a_{11} & \sqrt{2}(a_{44}-a_{11})+a_{23} \\ & & & & & \frac{-3a_{11}+a_{22}}{2} - a_{12} + 2a_{44} \end{pmatrix}.$$

HOMOGENIZED WAVE PROPAGATION (G. ROSI)

With explicit microstructure:

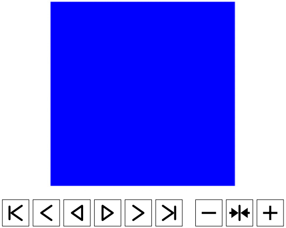


(a) Low frequency

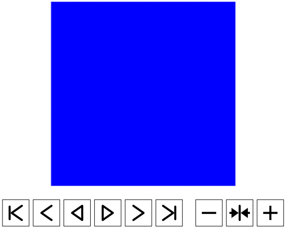


(b) High frequency

Once homogenized (c.f. talk of N. Bochud):



(c) Low frequency



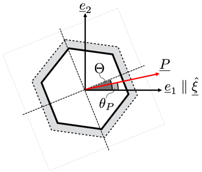
(d) High frequency

Pseudo cloaking effect in architectured materials [7]

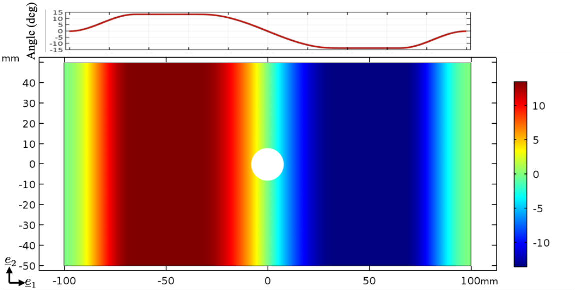
In condensed form

$$\underset{\approx}{\mathrm{A}}^{D_6}(\Theta) = \underset{\approx}{\mathrm{A}}^{O(2)} + a_D \underset{\approx}{\mathrm{A}}(\Theta)$$

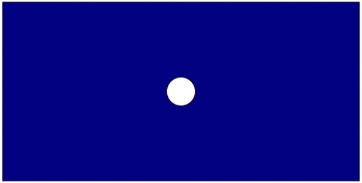
Schematic representation of the angles involved:



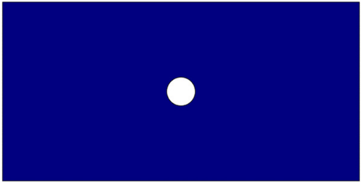
Distribution of the material orientation angle $\Theta_{opt}(x_1)$ within a sample



Application: Pseudo cloaking effect in architectured materials [7]



(a) $\theta = 0$



(b) θ is optimized

Next step

Deshomogenization of the optimal solution

CONCLUSION

1. set the difference between elastic tensors and elastic materials;
2. describe the elastic material domain in terms of invariants of the integrity basis;
3. investigate effects associated with weak scale separation;
4. extend the classical approach to higher order elasticity.

Connection with other topics of the project

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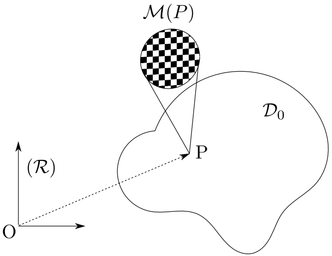
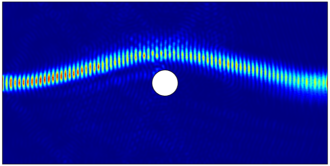
- **COMECH**: Topological optimization,...
- **BIO**: Growth and remodelling of biomaterials,...
- **NLS**: Control of smart materials, controlled instabilities, ...

Optimal design

- Response to specifications;
- Achieving non standard properties;
- Controlling wave propagation;
- ...





PERSPECTIVES

Extension/ Generalization

- Geometrization of other constitutive laws (2D setting);
  - General algorithm for computing integrity basis;
  - Piezo-electricity (third-order tensor);
  - Flexo-electricity (fourth-order);
  - Strain-gradient elasticity (Fifth- and sixth-order)
  - ...
- Extension to 3D elasticity
  - Cubic, Transverse isotropy....Full case
  - Reformulation of the Walpole classical table;
- Optimal Design
  - Exploring exotic anisotropic classes;
  - Deshomogenization problem;
  - Yield functions...

OUTLINE

N. Auffray, R. Bouchet, and Y. Bréchet. "Derivation of anisotropic matrix for bi-dimensional strain-gradient elasticity behavior." In: *International Journal of Solids and Structures* 46.2 (2009), pp. 440–454.

N. Auffray, J. Dirrenberger, and G. Rosi. "A complete description of bi-dimensional anisotropic strain-gradient elasticity". In: *International Journal of Solids and Structures* 69-70 (2015), pp. 195–206.

N. Auffray, B. Kolev, and M. Olive. "Handbook of bidimensional tensors: Part I: Decomposition and symmetry classes". In: *Mathematics and Mechanics of Solids* (2016), p. 1081286516649017.

B. Desmorat and N. Auffray. "Space of 2D elastic materials: a geometric journey". In: *Continuum Mechanics and Thermodynamics* 31.4 (2019), pp. 1205–1229.

M. Poncelet et al. "An experimental evidence of the failure of Cauchy elasticity for the overall modeling of a non-centro-symmetric lattice under static loading". In: *International Journal of Solids and Structures* (Accepted for publication).

G. Rosi and N. Auffray. "Anisotropic and dispersive wave propagation within strain-gradient framework". In: *Wave Motion* 63 (2016), pp. 120–134.

G. Rosi and N. Auffray. "Continuum modelling of frequency dependent acoustic beam focusing and steering in hexagonal lattices". In: *European Journal of Mechanics-A/Solids* Accepted (2019).

G. Rosi, L. Placidi, and N. Auffray. "On the validity range of strain-gradient elasticity: a mixed static-dynamic identification procedure". In: *European Journal of Mechanics-A/Solids* 69 (2018), pp. 179–191.