Dispersion of plane wave propagation in periodic poroelastic materials:

A comparison between Bloch-based and homogenization approaches

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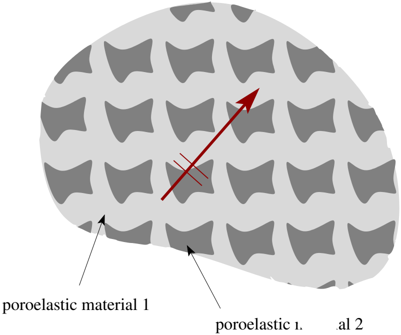
Introduction

Wave propagation in periodic poroelastic media

- Acoustic materials
- Biological tissues
- Geophysics

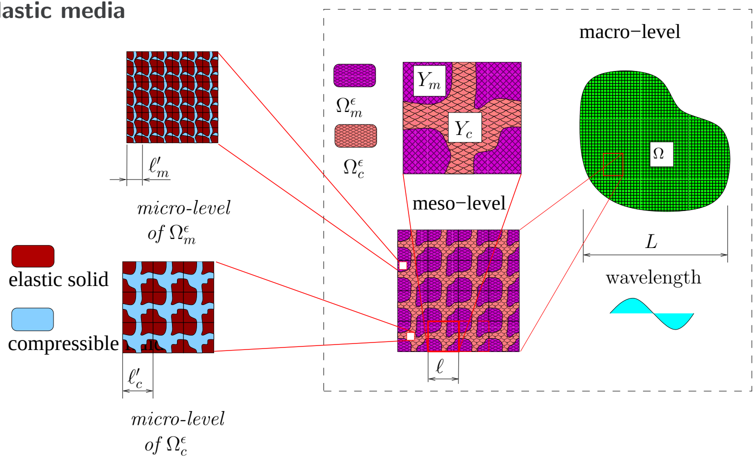


Objective

- Models for studying effective dynamic behavior
  - Two situations: low and high contrasts
  - Asymptotic homogenization technique
  - Numerically put in evidence the validity of the derived models

Introduction

Periodic poroelastic media



Only passage mesoscopic - macroscopic is interested

Equations at the mesoscale

Poroelastic equations in frequency domain: Biot's model [Biot, 1956]

- Dynamic equations

$$-\omega^2 \bar{\rho} \boldsymbol{u} + \mathrm{i}\omega \rho^f \boldsymbol{w} - \nabla \cdot \boldsymbol{\sigma} = \boldsymbol{0},$$
$$-\omega^2 \rho^f \boldsymbol{u} + [\boldsymbol{k}(\omega)]^{-1} \boldsymbol{w} + \nabla p = \boldsymbol{0},$$

- Constitutive equations

$$\boldsymbol{\sigma} = \mathbb{D}\boldsymbol{\epsilon}(\boldsymbol{u}) - \boldsymbol{\alpha} p$$
$$p = -M[\boldsymbol{\alpha} : \boldsymbol{e}(\boldsymbol{u}) + (\mathrm{i}\omega)^{-1} \nabla \cdot \boldsymbol{w}]$$

where

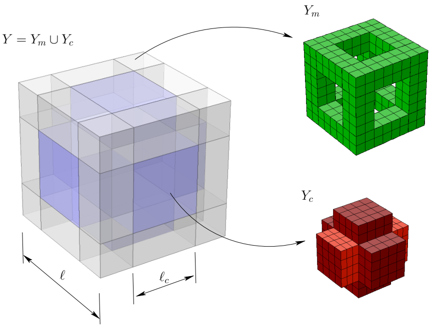
- $\boldsymbol{u}$: solid displacement
- $\boldsymbol{w}$: fluid effective velocity
  $\boldsymbol{w} = (\mathrm{i}\omega)\phi(\boldsymbol{u}^f - \boldsymbol{u}^s)$
- $\boldsymbol{\sigma}$: total stress tensor
- p: interstitial fluid pressure

- ρ^f, $\bar{\rho}$: mass densities
- $\mathbb{D}$: drained elastic tensor
- $\boldsymbol{\alpha}$, M: Biot's coupling constants
- $\boldsymbol{k}(\omega)$: dynamic permeability tensor

(Y-periodic)

Homogenized models

Low contrast vs high contrast



Low contrast

$$\bar{\rho}^{\varepsilon} = \chi_m(y)\bar{\rho}_m(y) + \chi_c(y)\bar{\rho}_c(y)$$

$$\mathbb{D}^{\varepsilon} = \chi_m(y)\mathbb{D}_m(y) + \chi_c(y)\widehat{\mathbb{D}}_c(y)$$

$$\boldsymbol{\alpha}^{\varepsilon} = \chi_m(y)\boldsymbol{\alpha}_m(y) + \chi_c(y)\boldsymbol{\alpha}_c(y)$$

$$\boldsymbol{K}^{\varepsilon} = \chi_m(y)\widehat{\boldsymbol{K}}_m(y) + \chi_c(y)\boldsymbol{K}_c(y)$$

High contrast

- Y_m is much more stiffer than Y_c
- Y_c is much more permeable than Y_m

$$\bar{\rho}^{\varepsilon} = \chi_m(y)\bar{\rho}_m(y) + \chi_c(y)\bar{\rho}_c(y)$$

$$\mathbb{D}^{\varepsilon} = \chi_m(y)\mathbb{D}_m(y) + \varepsilon^2\chi_c(y)\widehat{\mathbb{D}}_c(y)$$

$$\boldsymbol{\alpha}^{\varepsilon} = \chi_m(y)\boldsymbol{\alpha}_m(y) + \varepsilon\chi_c(y)\boldsymbol{\alpha}_c(y)$$

$$\boldsymbol{K}^{\varepsilon} = \varepsilon^2\chi_m(y)\widehat{\boldsymbol{K}}_m(y) + \chi_c(y)\boldsymbol{K}_c(y)$$

Homogenized model: low contrast case

Macroscopic wave equations

Macroscopic equations of effective medium

$$-\omega^2 \boldsymbol{\mathcal{M}} \boldsymbol{u}^0 - \nabla \cdot [\mathcal{D} \boldsymbol{e}(\boldsymbol{u}^0) - \boldsymbol{\mathcal{A}} p^0] - \mathrm{i}\omega \rho^f \boldsymbol{\mathcal{K}} \nabla p^0 = 0$$
$$\boldsymbol{\mathcal{A}} : \mathrm{i}\omega \boldsymbol{e}(\boldsymbol{u}^0) + \omega^2 \rho^f \nabla \cdot (\boldsymbol{\mathcal{K}} \boldsymbol{u}^0) - \nabla \cdot (\boldsymbol{\mathcal{K}} \nabla p^0) + \mathrm{i}\omega \mathcal{Q} p^0 = 0$$

similar to Biot's model!

- **Frequency-independent** effective properties

$$\mathcal{D}_{ijkl} = \int_Y [\mathbb{D}(y) \boldsymbol{e}_y(\boldsymbol{\chi}^{kl} + \boldsymbol{\Pi}^{kl})] : \boldsymbol{e}_y(\boldsymbol{\chi}^{ij} + \boldsymbol{\Pi}^{ij})$$
$$\mathcal{A}_{ij} = \int_Y \alpha_{ij}(y) - \int_Y [\mathbb{D}(y) \boldsymbol{e}_y(\boldsymbol{\chi} *)] : \boldsymbol{e}(\boldsymbol{\Pi}^{ij})$$
$$\mathcal{Q} = \int_Y M^{-1}(y)$$

- **Frequency-dependent** effective properties

$$\mathcal{K}_{ij} = -\mathrm{i}\omega \int_Y \left[\boldsymbol{K}(\omega, y) \nabla_y (\theta^i - \frac{1}{\mathrm{i}\omega} y_i) \right] \cdot \nabla_y y_j$$
$$\mathcal{M}_{ij}(\omega) = \int_Y \bar{\rho} \delta_{ij} - \mathrm{i}\omega \left(\rho^f \right)^2 \mathcal{K}_{ij}(\omega)$$

Homogenized model: high contrast case

Macroscopic wave equations

Macroscopic equations

$$-\omega^2 \boldsymbol{\mathcal{M}} \boldsymbol{u}^0 - \nabla \cdot [\boldsymbol{\mathcal{D}} \boldsymbol{e}(\boldsymbol{u}^0) - \boldsymbol{\mathcal{A}} p^0] - \mathrm{i}\omega\rho^f \boldsymbol{\mathcal{K}} \nabla p^0 + \mathrm{i}\omega \boldsymbol{\mathcal{C}} p^0 = 0$$

$$\boldsymbol{\mathcal{A}} : \mathrm{i}\omega \boldsymbol{e}(\boldsymbol{u}^0) + \omega^2 \rho^f \nabla \cdot (\boldsymbol{\mathcal{K}} \boldsymbol{u}^0) - \nabla \cdot (\boldsymbol{\mathcal{K}} \nabla p^0) + \mathrm{i}\omega \mathcal{Q} p^0 + \boldsymbol{\mathcal{C}} \cdot \boldsymbol{u}^0 = 0$$

- $\boldsymbol{\mathcal{D}}$, $\boldsymbol{\mathcal{A}}$ only depend on the response of Y_m and are frequency-independent
- $\boldsymbol{\mathcal{K}}$ only depends on the response of Y_c and is frequency-dependent
- $\boldsymbol{\mathcal{C}}$ depends on the response of Y_c and due to the heterogeneity of Biot's stress coupling coefficient

Details given in [Rohan, Naili, Nguyen, Computers & Structures (2017)

Validity of homogenized models

Validation strategies

FE simulation of real structures with real microstructure details

- can be done using a conventional FE analysis
- allows to directly capture local solution
- allows to consider boundary-value problems
- difficult to determine phase velocity and attenuation
- very high computational cost is required

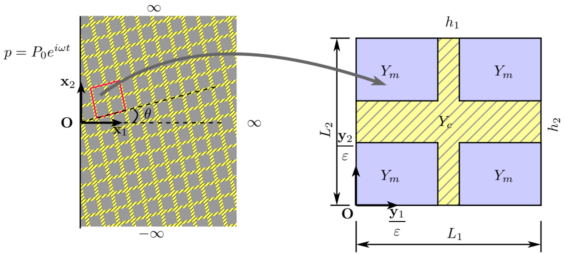
Floquet-Bloch analysis

- allows to directly compute the phase velocities and attenuation
- FE analysis of only one REV is needed
- no works have been found for poroelastic problem

Numerical examples

Half-space with orthogonal mesostructure : case of rigid solid phase

Geometry and physical parameters



- $L_1 = L_2 = 0.01$ m
- $h_1 = 0.4 L_1$
- $h_2 = 0.25 L_2$.
- κ_0^m variable
- $\kappa_0^c = 10^{-6}$ m^2/(Pa.s)
- $k^f = 2.25$ GPa
- $\eta = 10^{-3}$ Pa.s^{-1}

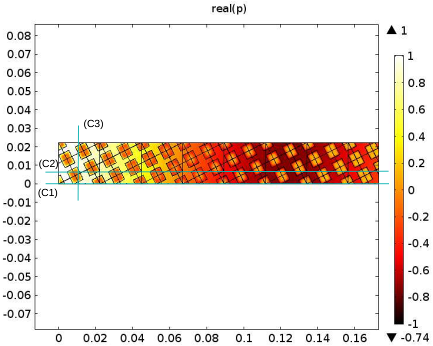
Objective

- comparison of solution obtained with low and high contrast homogenized model with FE solution
- check the domain of validity of each model.

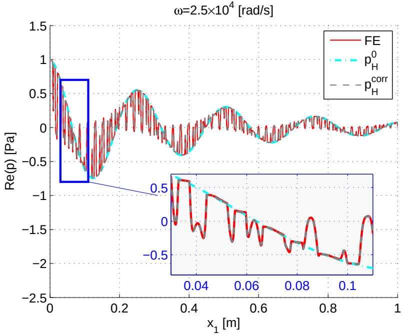
Numerical examples

Half-space with orthogonal mesostructure : case of rigid solid phase

Finite element solution

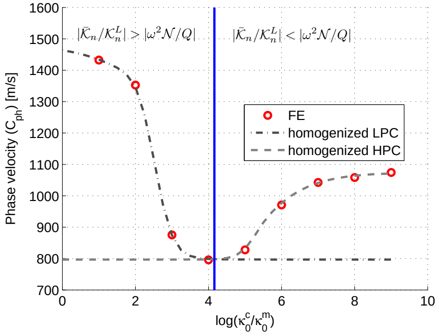


Validation of high contrast model

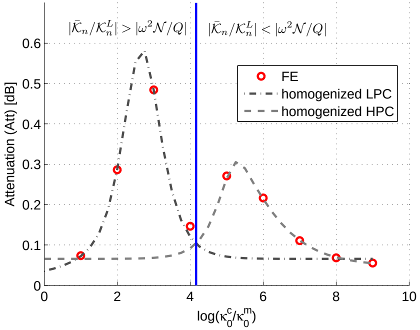


Numerical examples

Phase velocity



Attenuation



Wave dispersion analysis using Bloch-Floquet method

Governing equations

Biot's poroelastic equations

$$-\omega^2\rho\boldsymbol{u} - \omega^2\rho^f\mathbf{w} - \nabla\cdot\boldsymbol{\sigma} = \mathbf{0}$$
$$-\omega^2\rho^f\boldsymbol{u} - \omega^2\tilde{\mathbf{a}}\mathbf{w} + \nabla p = \mathbf{0},$$

&

$$\boldsymbol{\sigma} = \mathbb{C}:\boldsymbol{\epsilon} - \boldsymbol{\alpha}p,$$
$$p = -M\left(\boldsymbol{\alpha}:\boldsymbol{\epsilon} + \nabla\cdot\mathbf{w}\right),$$

Plane wave problem

- **Homogeneous media**

$$\boldsymbol{u}(\boldsymbol{x},\omega) = \boldsymbol{U}(\omega)(\boldsymbol{x})e^{-\mathrm{i}k\boldsymbol{n}\cdot\boldsymbol{x}}, \qquad \mathbf{w}(\boldsymbol{x},\omega) = \mathbf{W}(\omega)(\boldsymbol{x})e^{-\mathrm{i}k\boldsymbol{n}\cdot\boldsymbol{x}},$$

$\Rightarrow$ eigenvalue problem providing 4 complex wavenumbers k: 2 (quasi)-compresionnal waves (fast and slow) and 2 (quasi)-shear waves

- **Periodic media:** Bloch ansatz

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{U}(\boldsymbol{x},k)e^{-\mathrm{i}k\boldsymbol{n}\cdot\boldsymbol{x}}, \qquad \mathbf{w}(\boldsymbol{x}) = \mathbf{W}(\boldsymbol{x},k)e^{-\mathrm{i}k\boldsymbol{n}\cdot\boldsymbol{x}},$$

where $\boldsymbol{U}(\boldsymbol{x},k)$, $\mathbf{W}(\boldsymbol{x},k)$: Ω_E-periodic functions

Wave dispersion analysis using Bloch-Floquet method

Finite element formulation

Weak formulation

$$-\omega^2 \int_\Omega \bar{\boldsymbol{U}} \cdot \left(\rho \boldsymbol{U} + \rho^f \boldsymbol{W}\right) + \int_\Omega \bar{\boldsymbol{\mathcal{E}}} : \boldsymbol{\Sigma} + \mathrm{i}k \int_\Omega \left(\bar{\boldsymbol{\mathcal{E}}}_n : \boldsymbol{\Sigma} - \bar{\boldsymbol{\mathcal{E}}} : \boldsymbol{\Sigma}_n\right) + k^2 \int_\Omega \bar{\boldsymbol{\mathcal{E}}}_n : \boldsymbol{\Sigma}_n = \mathbf{0}$$

$$-\omega^2 \int_\Omega \bar{\mathbf{W}} \cdot \left(\rho^f \boldsymbol{U} + \tilde{\boldsymbol{a}} \mathbf{W}\right) - \int_\Omega (\nabla \cdot \bar{\mathbf{W}}) P - \mathrm{i}k \int_\Omega \left[(\boldsymbol{n} \cdot \bar{\mathbf{W}}) P - (\nabla \cdot \bar{\mathbf{W}}) P_n\right] - k^2 \int_\Omega (\boldsymbol{n} \cdot \bar{\mathbf{W}}) P_n = \mathbf{0}$$

Finite element equation

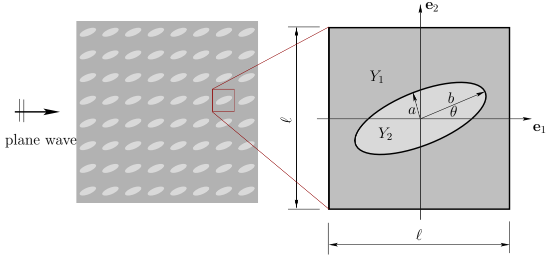
$$\left(-\omega^2 \mathbf{A}_0(\omega) + \mathbf{A}_1 + \mathrm{i}k\mathbf{A}_2 + k^2 \mathbf{A}_3\right) \mathbf{V} = \mathbf{0}$$

$\Rightarrow$ eigenvalue problem to determine the **dispersion relation** $\omega - k$

- viscous term is present due to permeabilty $\Rightarrow$ k is not real but complex [Collet *et al*, 2011]
- $\mathbf{A}$ is not constant w.r.t $\omega \Rightarrow k$ quadratic eigenvalue problem for k for each value of ω

Numerical results

Wave dispersion of a plane strain 2D poroelastic problem



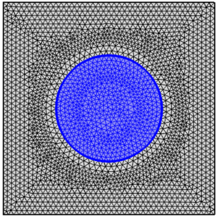
Physical properties

		Rock (Y_1)	Sandstone (Y_2)
ρ	kg.m^{-3}	2650	2650
ϕ	-	0.15	0.36
K^b	-	12.7	1.37
μ^b	GPa	20.3	0.82
α	-	0.6825	0.9658
μ	GPa	12.503	5.7096
k_0	m^2	0.1×10^{-12}	1.6×10^{-12}
ρ^f	kg.m^{-3}	1000	1000
K^f	GPa	2.25	2.25
η	Pa.s	10^{-3}	10^{-3}
a_∞	-	1	2.8

Comparison of wave dispersion obtained from homogenized model and Bloch analysis

Numerical examples

Floquet-Bloch analysis of a REV with circular inclusion



- $\ell = 1$ mm
- $a = b = 0.4$ mm

Eigenvalues obtained by Floquet-Bloch analysis

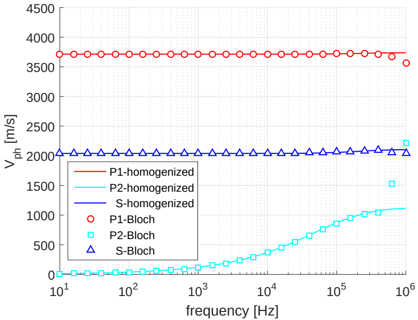
	$f = 10^4$ [Hz]	$f = 10^5$ [Hz]	$f = 3.98 \times 10^5$ [Hz]	$f = 10^6$ [Hz]
$\kappa^{(1)}$	**16.91 − 0.006i**	**168.89 − 0.435i**	**673.3 − 1.99i**	−4534.02 − 90.77i
$\kappa^{(2)}$	**30.71 − 0.057i**	**304.74 − 3.77i**	**1197.5 − 14.4i**	**1763.65 − 92.09i**
$\kappa^{(3)}$	**171.58 − 162.75i**	**729.63 − 408.67i**	**2398.8 − 587.26i**	−3216.03 − 663.90i
$\kappa^{(4)}$	499.93 − 867.014i	1298 − 1385.53i	−3894.4 − 588.66i	**3083.97 − 664.76i**
$\kappa^{(5)}$	656.25 − 890.34i	423.24 − 1585.8i	3572.7 − 1874.5i	**2842.23 − 1350.97i**

- k selected basing on its sign and value in order to get **less attenuated** and **propagative** modes

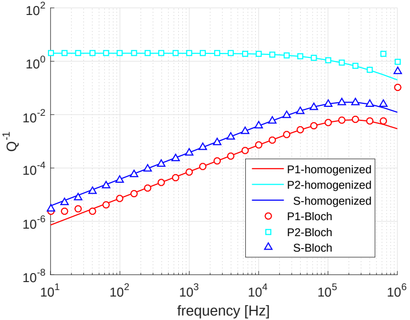
Numerical examples

Validation of homogenized model

Phase velocity

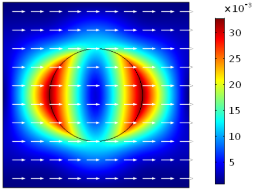


Attenuation

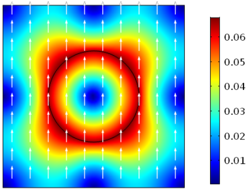


Numerical results

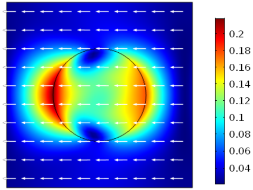
Bloch analysis



(a) Mode P_1, $f = 10^5$Hz

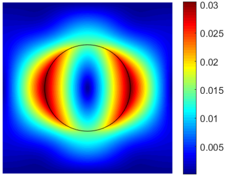


(b) Mode S, $f = 10^5$Hz

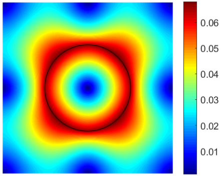


(c) Mode P_2, $f = 10^5$Hz

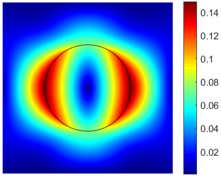
Homogenized model



(d) Mode P_1, $f = 10^5$Hz



(e) Mode S, $f = 10^5$Hz

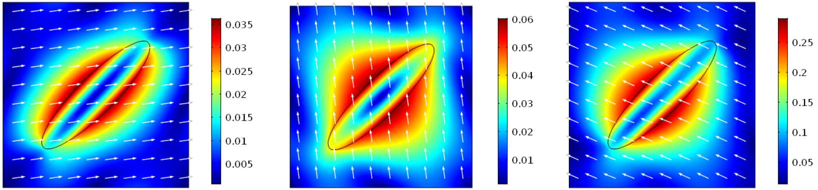


(f) Mode P_2, $f = 10^5$Hz

Numerical results

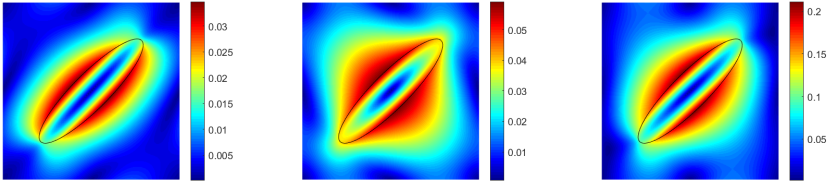
Elliptical inclusion

Bloch analysis



(g) Mode P1, $f = 10^5$ Hz (h) Mode S, $f = 10^5$ Hz (i) Mode P2, $f = 10^5$ Hz

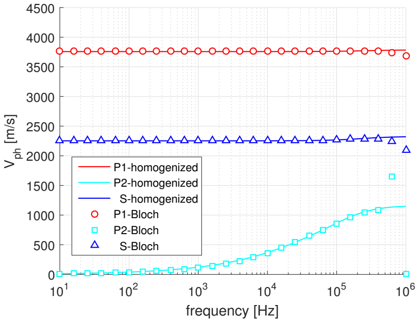
Homogenized mod



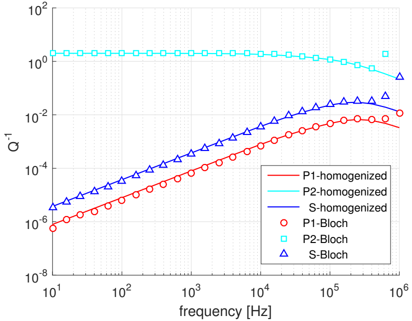
(j) Mode P1, $f = 10^5$ Hz (k) Mode S, $f = 10^5$ Hz (l) Mode P2, $f = 10^5$ Hz

Numerical examples
Validation of homogenized model

Phase velocity



Attenuation



Conclusions

- dynamic reponse of periodic medium can efficiently be predicted using the homogenized model
- the choice of appropriate model depends on the contrast of poroelastic and permeability properties
- finite element formulations have been derived for computing Floquet-Bloch eigenvalues of poroelastic media
- validation of time-domain responses has not been done
- for more details :
  - Nguyen *et al Int J Eng Sci*, 2016,
  - Rohan *etal Computers & Structures*, 2017,
  - Rohan *etal ZAMM*, 2018

Thank you for your attention